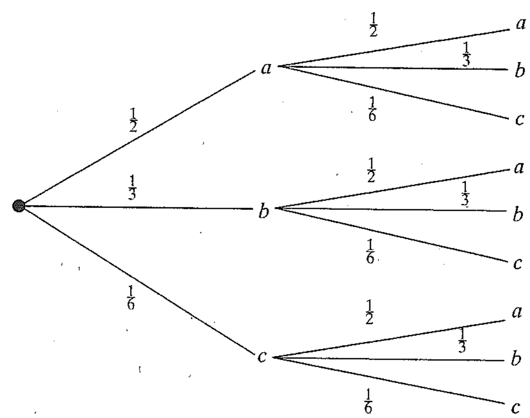


Fig. 4-7

Solved Problems

CONDITIONAL PROBABILITY

4.1. Three fair coins, a penny, a nickel, and a dime, are tossed. Find the probability p that they are all heads if: (a) the penny is heads, (b) at least one of the coins is heads, (c) the dime is tails.

The sample space has eight elements:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(a) If the penny is heads, the reduced sample space is $A = \{HHH, HHT, HTH, HTT\}$. All coins are heads in only 1 of the 4 cases; hence $p = 1/4$.

(b) If one or more of the coins is heads, the reduced sample space is

$$B = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$$

All coins are heads in only 1 of the 7 cases; hence $p = 1/7$.

(c) If the dime (third coin) is tails, the reduced sample space is $C = \{HHT, HTT, THT, TTT\}$. None contains all heads; hence $p = 0$.

4.2. A billiard ball is drawn at random from a box containing 15 billiard balls numbered 1 to 15, and the number n is recorded.

(a) Find the probability p that n exceeds 10.

(b) If n is even, find the probability p that n exceeds 10.

(a) The n can be one of the 5 numbers, 11, 12, 13, 14, 15. Hence $p = 5/15 = 1/3$.

(b) The reduced sample space E consists of the 7 even numbers, that is, $E = \{2, 4, 6, 8, 10, 12, 14\}$. Of these, only 2, 12, and 14, exceed 10. Hence $p = 2/7$.

4.3. A pair of fair dice is thrown. Find the probability p that the sum is 10 or greater if:

(a) 5 appears on the first die, (b) 5 appears on at least one die.

Figure 3-3 shows the 36 ways the pair of dice can be thrown.

(*a*) If 5 appears on the first die, the reduced sample space A has six elements:

$$A = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

The sum is 10 or more on 2 of the 6 outcomes, $(5,5)$ and $(5,6)$. Thus, $p = 2/6 = 1/3$.

(*b*) If 5 appears on at least one die, the reduced sample space B has 11 elements:

$$B = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5)\}$$

The sum is 10 or more on 3 of the 11 outcomes, $(5,5)$, $(5,6)$, and $(6,5)$. Thus, $p = 3/11$.

4.4. In a certain college, 25 percent of the students failed mathematics, 15 percent failed chemistry, and 10 percent failed both mathematics and chemistry. A student is selected at random.

(*a*) If the student failed chemistry, what is the probability that he or she failed mathematics?

(*b*) If the student failed mathematics, what is the probability that he or she failed chemistry?

(*c*) What is the probability that the student failed mathematics or chemistry?

(*d*) What is the probability that the student failed neither mathematics nor chemistry?

(*a*) We seek $P(M|C)$, the probability that the student failed mathematics, given that he or she failed chemistry. By definition,

$$P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{10}{15} = \frac{2}{3}$$

(*b*) We seek $P(C|M)$, the probability that the student failed chemistry, given that he or she failed mathematics. By definition,

$$P(C|M) = \frac{P(M \cap C)}{P(M)} = \frac{0.10}{0.25} = \frac{10}{25} = \frac{2}{5}$$

(*c*) By the addition rule (Theorem 3.6),

$$P(M \cup C) = P(M) + P(C) - P(M \cap C) = 0.25 + 0.15 - 0.10 = 0.30$$

(*d*) Students who failed neither mathematics nor chemistry form the complement of the set $M \cup C$, that is, form the set $(M \cup C)^c$. Hence

$$P((M \cup C)^c) = 1 - P(M \cup C) = 1 - 0.30 = 0.70$$

4.5. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability p that: (*a*) the sum is 6, (*b*) an ace appears, (*c*) the sum is 4 or less.

There are 36 ways the pair of dice can be thrown (Fig. 3-3) and 6 of them, $(1,1), (2,2), \ldots, (6,6)$, have the same numbers. Thus, the reduced sample space E will consist of $36 - 6 = 30$ elements.

(*a*) The sum 6 can appear in 4 ways: $(1,5)$, $(2,4)$, $(4,2)$, $(5,1)$. [We cannot include $(3,3)$ since the numbers must be different.] Thus, $p = 4/30 = 2/15$.

(*b*) An ace can appear in 10 ways: $(1,2), (1,3), \ldots, (1,6)$ and $(2,1), (3,1), \ldots, (6,1)$. [We cannot include $(1,1)$ since the numbers must be different.] Thus, $p = 10/30 = 1/3$.

(*c*) The sum is 4 or less in 4 ways: $(3,1)$, $(1,3)$, $(2,1)$, $(1,2)$. [We cannot include $(1,1)$ and $(2,2)$ since the numbers must be different.] Thus, $p = 4/30 = 2/15$.

4.6. Let A and B be events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find:

(*a*) $P(A|B)$ and $P(B|A)$, (*b*) $P(A \cup B)$, (*c*) $P(A^c)$ and $P(B^c)$.

(*a*) By definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.3} = \frac{2}{3}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.6} = \frac{1}{3}$$

(*b*) By the addition rule (Theorem 3.6),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.3 - 0.2 = 0.7$$

(*c*) By the complement rule,

$$P(A^c) = 1 - P(A) = 1 - 0.6 = 0.4 \quad \text{and} \quad P(B^c) = 1 - P(B) = 1 - 0.3 = 0.7$$

4.7. Consider the data in Problem 4.6. Find: (*a*) $P(A^c|B^c)$, (*b*) $P(B^c|A^c)$.

First compute $P(A^c \cap B^c)$. By DeMorgan's law, $(A \cup B)^c = A^c \cap B^c$. Hence, by the complement rule,

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

(*a*) $P(A^c|B^c) = \dfrac{P(A^c \cap B^c)}{P(B^c)} = \dfrac{0.3}{0.7} = \dfrac{3}{7}$

(*b*) $P(B^c|A^c) = \dfrac{P(A^c \cap B^c)}{P(A^c)} = \dfrac{0.3}{0.4} = \dfrac{3}{4}$

4.8. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$, and $P(A \cup B) = \frac{3}{4}$. Find $P(A|B)$ and $P(B|A)$.

First find $P(A \cap B)$ using the addition rule that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. We have

$$\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(A \cap B) \quad \text{or} \quad P(A \cap B) = \frac{1}{4}$$

Now use the definition of conditional probability to get

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{5/8} = \frac{2}{5} \quad \text{and} \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/8} = \frac{2}{3}$$

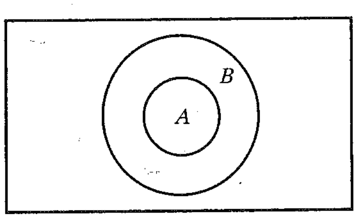
4.9. Find $P(B|A)$ if: (*a*) A is a subset of B, (*b*) A and B are mutually exclusive (disjoint). [Assume $P(A) > 0$.]

(*a*) If A is a subset of B [as pictured in Fig. 4-8(*a*)], then whenever A occurs, B must occur; hence $P(B|A) = 1$. Alternately, if A is a subset of B, then $A \cap B = A$; hence

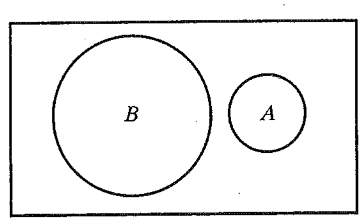
$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

(*b*) If A and B are mutually exclusive, that is, disjoint [as pictured in Fig. 4-8(*b*)], then whenever A occurs, B cannot occur; hence $P(B|A) = 0$. Alternately, if A and B are disjoint, then $A \cap B = \varnothing$; hence

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\varnothing)}{P(A)} = \frac{0}{P(A)} = 0$$



(*a*) $A \subseteq B$.



(*b*) $A \cap B = \varnothing$.

Fig. 4-8

4.10. Let E be an event for which $P(E) > 0$. Show that the conditional probability function $P(*|E)$ satisfies the axioms of a probability space, that is

[P_1] For any event A, we have $P(A|E) \geq 0$.

[P_2] For any certain event S, we have $P(S|E) = 1$.

[P_3] For any two disjoint events A and B, we have

$$P(A \cup B|E) = P(A|E) + P(B|E)$$

[P_3'] For any infinite sequence of mutually disjoint events $A_1, A_2, \ldots$, we have

$$P(A_1 \cup A_2 \cup \cdots |E) = P(A_1|E) + P(A_2|E) + \cdots$$

(*a*) We have $P(A \cap E) \geq 0$ and $P(E) > 0$; hence

$$P(A|E) = \frac{P(A \cap E)}{P(E)} \geq 0$$

Thus, [P_1] holds.

(*b*) We have $S \cap E = E$; hence

$$P(S|E) = \frac{P(S \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1$$

Thus, [P_2] holds.

(*c*) If A and B are disjoint events, then so are $A \cap E$ and $B \cap E$. Furthermore,

$$(A \cup B) \cap E = (A \cap E) \cup (B \cap E)$$

Hence,

$$P[(A \cup B) \cap E] = P[(A \cap E) \cup (B \cap E)] = P(A \cap E) + P(B \cap E)$$

Therefore

$$P(A \cup B|E) = \frac{P[(A \cup B) \cap E]}{P(E)} = \frac{P(A \cap E) + P(B \cap E)}{P(E)}$$

$$= \frac{P(A \cap E)}{P(E)} + \frac{P(B \cap E)}{P(E)} = P(A|E) + P(B|E)$$

Thus, [P_3] holds.

(*d*) [Similar to (*c*).] If $A_1, A_2, \ldots$ are mutually disjoint events, then so are $A_1 \cap E, A_2 \cap E, \ldots$. Also, by the generalized distributive law,

$$(A_1 \cup A_2 \cup \cdots) \cap E = (A_1 \cap E) \cup (A_2 \cap E) \cup \cdots$$

Thus

$$P[(A_1 \cup A_2 \cup \cdots) \cap E] = P[(A_1 \cap E) \cup (A_2 \cap E) \cup \cdots]$$
$$= P(A_1 \cap E) + P(A_2 \cap E) + \cdots$$

Therefore

$$P(A_1 \cup P_2 \cup \cdots |E) = \frac{P[(A_1 \cup A_2 \cup \cdots) \cap E]}{P(E)}$$

$$= \frac{P(A_1 \cap E) + P(A_2 \cap E) + \cdots}{P(E)} = \frac{P(A_1 \cap E)}{P(E)} + \frac{P(A_2 \cap E)}{P(E)} + \cdots$$

$$= P(A_1|E) + P(A_2|E) + \cdots$$

Thus, [P_3'] holds.

MULTIPLICATION THEOREM

4.11. A class has 12 men and 4 women. Suppose 3 students are selected at random from the class. Find the probability p that they are all men.

The probability that the first student is a man is 12/16 since there are 12 men out of the 16 students. If the first student is a man, then the probability that the second student is a man is 11/15 since there are 11 men left out of the 15 students left. Finally, if the first 2 students are men, then the probability that the third student is a man is 10/14 since there are now only 10 men out of the 14 students left. Accordingly, by the Multiplication Theorem 4.2, the probability that all 3 are men is

$$p = \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{28}$$

Another Method: There are $C(16, 3) = 560$ ways to select 3 students out of 16 students, and $C(12, 3) = 220$ ways to select 3 men from the 12 men. Thus

$$p = \frac{220}{560} = \frac{11}{28}$$

A Third Method: Suppose the students are selected one after the other. Then there are $16 \cdot 15 \cdot 14$ ways to select 3 students, and there are $12 \cdot 11 \cdot 10$ ways to select the 3 men. Thus

$$p = \frac{16 \cdot 15 \cdot 14}{12 \cdot 11 \cdot 10} = \frac{11}{28}$$

4.12. A person is dealt 5 cards from an ordinary 52-card deck (Fig. 3-4). Find the probability p that they are all spades.

The probability that the first card is a spade is 13/52, that the second is a spade is 12/51, that the third is a spade is 11/50, and that the fourth is a spade is 10/48. (We assume in each case that the previous cards were spades.) Thus, by the Multiplication Theorem 4.2,

$$p = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{48} = \frac{33}{66{,}640} = 0.000\,49$$

Another Method: There are $C(52, 5)$ ways to select 5 cards from the 52-card deck, and $C(13, 5)$ ways to select 5 spades from the 13 spades. Thus

$$p = \frac{C(13, 5)}{C(52, 5)} \approx 0.000\,49$$

4.13. A box contains 7 red marbles and 3 white marbles. Three marbles are drawn from the box one after the other. Find the probability p that the first 2 are red and the third is white.

The probability that the first marble is red is 7/10 since there are 7 red marbles out of the 10 marbles. If the first marble is red, then the probability that the second marble is red is 6/9 since there are 6 red marbles out of the remaining 9 marbles. Finally, if the first 2 marbles are red, then the probability that the third marble is white is 3/8 since there are 3 white marbles out of the remaining 8 marbles in the box. Accordingly, by the Multiplication Theorem 4.2,

$$p = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{7}{40} = 0.175 = 17.5\%$$

4.14. Students in a class are selected at random, one after the other, for an examination. Find the probability p that the men and women in the class alternate if:

(a) the class consists of 4 men and 3 women, (b) the class consists of 3 men and 3 women.

(*a*) If the men and women are to alternate, then the first student must be a man. The probability that the first is a man is 4/7. If the first is a man, the probability that the second is a woman is 3/6 since there are 3 women out of the 6 students left. Continuing in this manner, we obtain that the probability that the third is a man is 3/5, the fourth is a woman is 2/4, that the fifth is a man is 2/3, that the sixth is a woman is 1/2, and that the last is a man is 1/1. Thus

$$p = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{35}$$

(*b*) There are two mutually exclusive cases: the first student is a man and the first is a woman. If the first student is a man, then, by the multiplication theorem, the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}$$

If the first student is a woman, then, by the multiplication theorem, the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}$$

Thus, $p = p_1 + p_2 = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$.

FINITE STOCHASTIC PROCESSES

4.15. Let X, Y, Z be three coins in a box. Suppose X is a fair coin, Y is two-headed, and Z is weighted so that the probability of heads is 1/3. A coin is selected at random and is tossed. (*a*) Find the probability that heads appears, that is, find $P(H)$. (*b*) If heads appears, find the probability that it is the fair coin X, that is, find $P(X|H)$. (*c*) If tails appears, find the probability it is the coin Z, that is, find $P(Z|T)$.

Construct the corresponding two-step stochastic tree diagram in Fig. 4-9(*a*).

(*a*) Heads appears along three of the paths; hence

$$P(H) = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\frac{1}{3} = \frac{11}{18}$$

(*b*) Note X and heads H appear only along the top path in Fig. 4-9(*a*); hence

$$P(X \cap H) = (1/3)(1/2) = 1/6 \quad \text{and so} \quad P(X|H) = \frac{P(X \cap H)}{P(H)} = \frac{1/6}{11/18} = \frac{3}{11}$$

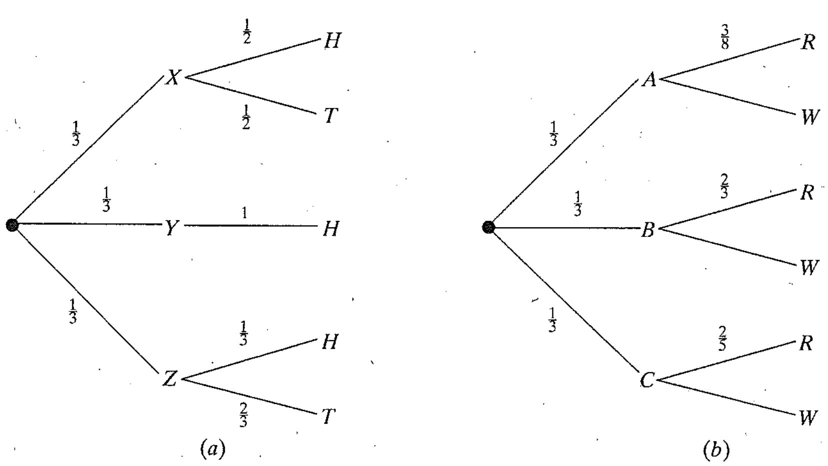


Fig. 4-9

(c) $P(T) = 1 - P(H) = 1 - 11/18 = 7/18$. Alternately, tails appears along two of the paths and so

$$P(T) = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{2}{3} = \frac{7}{18}$$

Note Z and tails T appear only along the bottom path in Fig. 4-9(a); hence

$$P(Z \cap T) = (1/3)(2/3) = 2/9 \quad \text{and so} \quad P(Z\,|\,T) = \frac{P(Z \cap T)}{P(T)} = \frac{2/9}{7/18} = \frac{4}{7}$$

4.16. Suppose the following three boxes are given:

Box A contains 3 red and 5 white marbles.
Box B contains 2 red and 1 white marbles.
Box C contains 2 red and 3 white marbles.

A box is selected at random, and a marble is randomly drawn from the box. If the marble is red, find the probability that it came from box A.

Construct the corresponding stochastic tree diagram as in Fig. 4-9(b). We seek $P(A\,|\,R)$, the probability that A was selected, given that the marble is red. Thus, it is necessary to find $P(A \cap R)$ and $P(R)$. Note that A and R only occur on the top path; hence $P(A \cap R) = (1/3)(3/8) = 1/8$. There are three paths leading to a red marble R; hence

$$P(R) = \frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{5} = \frac{173}{360} \approx 0.48$$

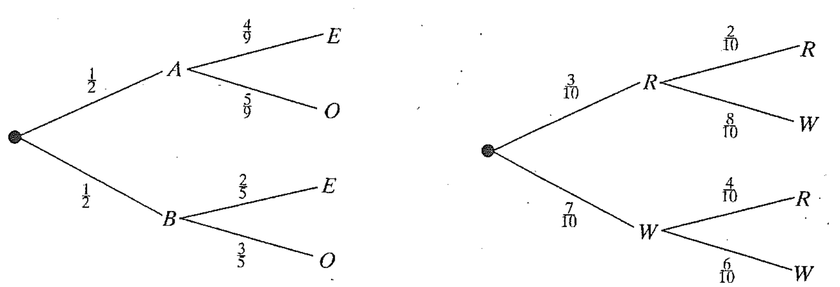
Thus

$$P(A\,|\,R) = \frac{P(A \cap R)}{P(R)} = \frac{1/8}{173/360} = \frac{45}{173} \approx 0.26$$

4.17. Box A contains 9 cards numbered 1 through 9, and box B contains 5 cards numbered 1 through 5. A box is selected at random, and a card is randomly drawn from the box. If the number is even, find the probability that the card came from box A.

Construct the corresponding stochastic tree diagram as in Fig. 4-10(a). We seek $P(A\,|\,E)$, the probability that A was selected, given that the number is even. Thus, it is necessary to find $P(A \cap E)$ and $P(E)$. Note that A and E only occur on the top path; hence $P(A \cap E) = (1/2)(4/9) = 2/9$. Two paths lead to an even number E; hence

$$P(E) = \frac{1}{2}\cdot\frac{4}{9} + \frac{1}{2}\cdot\frac{2}{5} = \frac{19}{45} \quad \text{and so} \quad P(A\,|\,E) = \frac{P(A \cap E)}{P(E)} = \frac{2/9}{19/45} = \frac{10}{19} \approx 0.53$$



(a) (b)

Fig. 4-10

4.18. A box contains 3 red marbles and 7 white marbles. A marble is drawn from the box and the marble is replaced by a marble of the other color. A second marble is drawn from the box.

(*a*) Find the probability p that the second marble is red.

(*b*) If both marbles were of the same color, find the probability p that they both were white.

Construct the corresponding stochastic tree diagram as in Fig. 4-10(*b*).

(*a*) Two paths lead to a red marble R; hence

$$p = \frac{3}{10} \cdot \frac{2}{10} + \frac{7}{10} \cdot \frac{4}{10} = \frac{17}{50} = 0.34$$

(*b*) Note that W appears twice only on the bottom path; hence $P(WW) = (7/10)(6/10) = 21/50$ is the probability that both were white. There are two paths, the top path and the bottom path, where the marbles are the same color. Thus

$$P(RR \text{ or } WW) = \frac{3}{10} \cdot \frac{2}{10} + \frac{7}{10} \cdot \frac{6}{10} = \frac{12}{25}$$

is the probability of the same color, the reduced sample space. Therefore

$$p = \frac{21/50}{12/25} = \frac{7}{8} = 0.875$$

4.19. A box contains a fair coin A and a two-headed coin B. A coin is selected at random and tossed twice.

(*a*) If heads appears both times, find the probability p that the coin is two-headed.

(*b*) If tails appears both times, find the probability p that the coin is two-headed.

Construct the corresponding stochastic tree diagram as in Fig. 4-11.

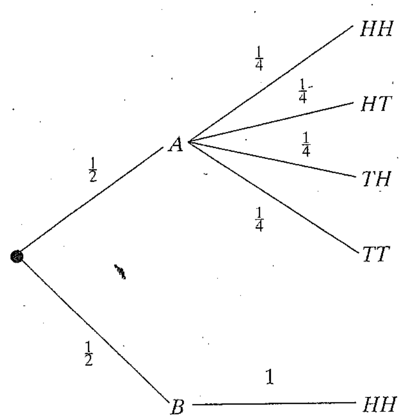


Fig. 4-11

(*a*) We seek $P(B|HH)$. Heads appears twice only in the top path and in the bottom path. Hence

$$P(HH) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 = \frac{5}{8}$$

On the other hand, $P(B \cap HH) = P(B) = \frac{1}{2}$. Thus

$$p = P(B|HH) = \frac{P(B \cap HH)}{P(B)} = \frac{1/2}{5/8} = \frac{4}{5}$$

(*b*) If tails appears then it could not be the two-headed coin B. Hence $p = 0$.

4.20. Suppose the following two boxes are given:

Box A contains 3 red and 2 white marbles.

Box B contains 2 red and 5 white marbles.

A box is selected at random; a marble is drawn and put into the other box; then a marble is drawn from the second box. Find the probability p that both marbles drawn are of the same color.

Construct the corresponding stochastic tree diagram as in Fig. 4-12. Note that this is a three-step stochastic process: (1) choosing a box, (2) choosing a marble, (3) choosing a second marble. Note that if box A is selected and a red marble R is drawn and put into box B, then box B will have 3 red marbles and 5 white marbles.

There are 4 paths which lead to 2 marbles of the same color; hence

$$p = \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{7}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{5}{7}\cdot\frac{1}{2} = \frac{901}{1680} \approx 0.536$$

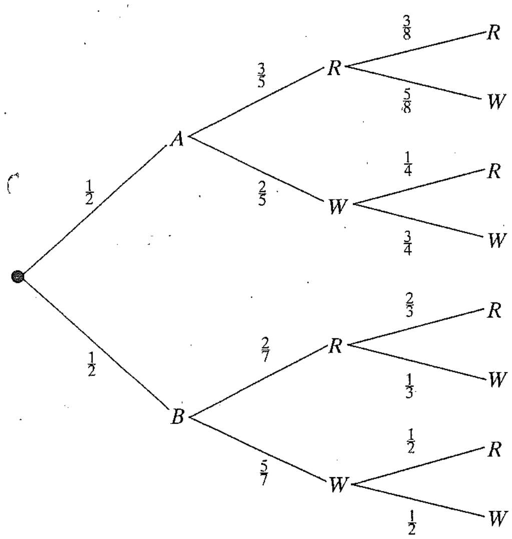


Fig. 4-12

LAW OF TOTAL PROBABILITY, BAYES' RULE

4.21. In a certain city, 40 percent of the people consider themselves Conservatives (C), 35 percent consider themselves to be Liberals (L), and 25 percent consider themselves to be Independents (I). During a particular election, 45 percent of the Conservatives voted, 40 percent of the Liberals voted and 60% of the Independents voted. Suppose a person is randomly selected.

(*a*) Find the probability that the person voted.

(*b*) If the person voted, find the probability that the voter is

(i) Conservative, (ii) Liberal, (iii) Independent.

(a) Let V denote the event that a person voted. We need $P(V)$. By the law of total probability,

$$P(V) = P(C)P(V|C) + P(L)P(V|L) + P(I)P(V|I)$$
$$= (0.40)(0.45) + (0.35)(0.40) + (0.25)(0.60) = 0.47$$

(b) Use Bayes' rule:

(i) $P(C|V) = \frac{P(C)P(V|C)}{P(V)} = \frac{(0.40)(0.45)}{0.47} = \frac{18}{47} \approx 38.3\%$

(ii) $P(L|V) = \frac{P(L)P(V|L)}{P(V)} = \frac{(0.35)(0.40)}{0.47} = \frac{14}{47} \approx 29.8\%$

(iii) $P(I|V) = \frac{P(I)P(V|I)}{P(V)} = \frac{(0.25)(0.60)}{0.47} = \frac{15}{47} \approx 31.9\%$

4.22. In a certain college, 4 percent of the men and 1 percent of the women are taller than 6 feet. Furthermore, 60 percent of the students are women. Suppose a randomly selected student is taller than 6 feet. Find the probability that the student is a woman.

Let A = {students taller than 6 feet}. We seek $P(W|A)$, the probability that a student is a woman, given that the student is taller than 6 feet. By Bayes' formula,

$$P(W|A) = \frac{P(W)P(A|W)}{P(W)P(A|W) + P(M)P(A|M)} = \frac{(0.60)(0.01)}{(0.60)(0.01) + (0.40)(0.04)} = \frac{3}{11}$$

4.23. Three machines A, B, and C produce, respectively, 40 percent, 10 percent, and 50 percent of the items in a factory. The percentage of defective items produced by the machines is, respectively, 2 percent, 3 percent, and 4 percent. An item from the factory is selected at random.

(a) Find the probability that the item is defective.

(b) If the item is defective, find the probability that the item was produced by:

(i) machine A,

(ii) machine B,

(iii) machine C.

(a) Let D denote the event that an item is defective. Then, by the law of total probability,

$$P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)$$
$$= (0.40)(0.02) + (0.10)(0.03) + (0.50)(0.04) = 0.031 = 3.1\%$$

(b) Use Bayes' formula to obtain

(i) $P(A|D) = \frac{P(A)P(D|A)}{P(D)} = \frac{(0.40)(0.02)}{0.031} = \frac{8}{31} \approx 25.8\%$

(ii) $P(B|D) = \frac{P(B)P(B|Y)}{P(D)} = \frac{(0.10)(0.03)}{0.031} = \frac{3}{31} \approx 9.7\%$

(iii) $P(C|D) = \frac{P(C)P(D|C)}{P(D)} = \frac{(0.50)(0.04)}{0.031} = \frac{20}{31} \approx 64.5\%$

4.24. Suppose a student dormitory in a college consists of:

(1) 40 percent freshmen of whom 15 percent are New York residents
(2) 25 percent sophomores of whom 40 percent are New York residents
(3) 20 percent juniors of whom 25 percent are New York residents
(4) 15 percent seniors of whom 20 percent are New York residents

A student is randomly selected from the dormitory.

(*a*) Find the probability that the student is a New York resident.

(*b*) If the student is a New York resident, find the probability that the student is a: (i) freshman, (ii) junior.

Let A, B, C, D denote, respectively, the set of freshmen, sophomores, juniors, and seniors, and let E denote the set of students who are New York residents.

(*a*) We find $P(E)$ by the law of total probability. We have:

$$P(E) = (0.40)(0.15) + (0.25)(0.40) + (0.20)(0.25) + (0.15)(0.20) = 0.06 + 0.10 + 0.05 + 0.03 = 0.24 = 24\%$$

(*b*) Use Bayes' formula to obtain:

$$\text{(i)}\quad P(A|E) = \frac{P(A)P(E|A)}{P(E)} = \frac{(0.40)(0.15)}{0.24} = \frac{6}{24} = 25\%$$

$$\text{(ii)}\quad P(C|E) = \frac{P(C)P(E|C)}{P(E)} = \frac{(0.20)(0.25)}{0.24} = \frac{5}{25} = 20\%$$

4.25. A box contains 10 coins where 5 coins are two-headed, 3 coins are two-tailed, and 2 are fair coins. A coin is chosen at random and tossed.

(*a*) Find the probability that a head appears.

(*b*) If a head appears, find the probability that the coin is fair.

Let X, Y, Z denote, respectively, the two-headed coins, the two-tailed coins, and the fair coins. Then $P(X) = 0.5$, $P(Y) = 0.3$, $P(Z) = 0.2$. Note $P(H|X) = 1$, that is, a two-headed coin must yield a head. Similarly, $P(H|Y) = 0$ and $P(H|Z) = 0.5$. Figure 4-13 is a stochastic tree (with the root at the top) describing the given data.

(*a*) By the law of total probability or by adding the probabilities of the three paths in Fig. 4-13 leading to H, we get

$$P(H) = (0.5)(1) + (0.3)(0) + (0.2)(0.5) = 0.6$$

(*b*) By Bayes' rule,

$$P(Z|H) = \frac{P(Z)P(H|Z)}{P(H)} = \frac{(0.2)(0.5)}{0.6} = \frac{1}{6} = 16.7\%$$

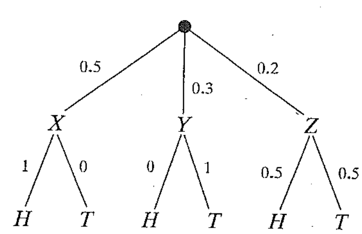


Fig. 4-13

INDEPENDENT EVENTS

4.26. Two men A and B fire at a target. Suppose $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{5}$ denote their probabilities of hitting the target. (We assume that the events A and B are independent.) Find the probability that:

(*a*) A does not hit the target.
(*b*) Both hit the target.
(*c*) One of them hits the target.
(*d*) Neither hits the target.

(*a*) By the complement rule,

$$P(\text{not } A) = P(A^c) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$$

(*b*) Since the events A and B are independent,

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

(*c*) By the addition rule (Theorem 3.6),

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \frac{7}{15}$$

(*d*) By DeMorgan's law, "neither A nor B" is the complement of $A \cup B$. [See Problem 3.1(*b*).] Hence

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{7}{15} = \frac{8}{15}$$

4.27. Box A contains 5 red marbles and 3 blue marbles and Box B contains 3 red and 2 blue. A marble is drawn at random from each box.

(*a*) Find the probability p that both marbles are red.
(*b*) Find the probability p that one is red and one is blue.

(*a*) The probability of choosing a red marble from A is $\frac{5}{8}$ and a red marble from B is $\frac{3}{5}$. Since the events are independent,

$$p = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$$

(*b*) There are two (mutually exclusive) events:

X: a red marble from A and a blue marble from B
Y: a blue marble from A and a red marble from B

We have

$$P(X) = \frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4} \quad \text{and} \quad P(Y) = \frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$$

Accordingly, since X and Y are mutually exclusive,

$$p = P(X) + P(Y) = \frac{1}{4} + \frac{9}{40} = \frac{19}{40}$$

4.28. Let A be the event that a man will live 10 more years, and let B be the event that his wife lives 10 more years. Suppose $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$. Assuming A and B are independent events, find the probability that, in 10 years

(*a*) Both will be alive. (*c*) Neither will be alive.
(*b*) At least one will be alive. (*d*) Only the wife will be alive.

(*a*) We seek $P(A \cap B)$. Since A and B are independent events,

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

(*b*) We seek $P(A \cup B)$. By the addition rule (Theorem 3.6),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{1}{2}$$

(*c*) By DeMorgan's law, "neither A nor B" is the complement of $A \cup B$. [Problem 3.1(*b*).] Hence

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{1}{2} = \frac{1}{2}$$

Alternately, we have $P(A^c) = \frac{3}{4}$ and $P(B^c) = \frac{2}{3}$; and, since A^c and B^c are independent,

$$P(A^c \cap B^c) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

(*d*) We seek $P(A^c \cap B)$. Since A^c and B are also independent,

$$P(A^c \cap B) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

4.29. Consider the following events for a family with children:

$$A = \{\text{children of both sexes}\}, \qquad B = \{\text{at most one boy}\}$$

(*a*) Show that A and B are independent events if a family has 3 children.
(*b*) Show that A and B are dependent events if a family has only 2 children.

(*a*) We have the equiprobable space $S = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$. Here

$$A = \{bbg, bgb, bgg, gbb, gbg, ggb\} \quad \text{and so} \quad P(A) = \frac{6}{8} = \frac{3}{4}$$

$$B = \{bgg, gbg, ggb, ggg\} \quad \text{and so} \quad P(B) = \frac{4}{8} = \frac{1}{2}$$

$$A \cap B = \{bgg, gbg, ggb\} \quad \text{and so} \quad P(A \cap B) = \frac{3}{8}$$

Since $P(A)P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$, A and B are independent.

(*b*) We have the equiprobable space $S = \{bb, bg, gb, gg\}$. Here

$$A = \{bg, gb\} \quad \text{and so} \quad P(A) = \frac{1}{2}$$

$$B = \{bg, gb, gg\} \quad \text{and so} \quad P(B) = \frac{3}{4}$$

$$A \cap B = \{bg, gb\} \quad \text{and so} \quad P(A \cap B) = \frac{1}{2}$$

Since $P(A)P(B) \neq P(A \cap B)$, A and B are dependent.

4.30. Three men A, B, C fire at a target. Suppose $P(A) = 1/6$, $P(B) = 1/4$, $P(C) = 1/3$ denote their probabilities of hitting the target. (We assume that the events that A, B, C hit the target are independent.)

(*a*) Find the probability p that they all hit the target.

(*b*) Find the probability p that they all miss the target.

(*c*) Find the probability p that at least one of them hits the target.

(*a*) We seek $P(A \cap B \cap C)$. Since A, B, C are independent events,

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{72} = 1.4\%$$

(*b*) We seek $P(A^c \cap B^c \cap C^c)$. We have $P(A^c) = 1 - P(A) = 5/6$. Similarly, $P(B^c = 3/4$ and $P(C^c) = 2/3$. Since A, B, C are independent events, so are A^c, B^c, C^c. Hence

$$P(A^c \cap B^c \cap C^c) = P(A^c) \cdot P(B^c) \cdot P(C^c) = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{5}{12} = 41.7\%$$

(*c*) Let D be the event that one or more of them hit the target. Then D is the complement of the event $A^c \cap B^c \cap C^c$, that they all miss the target. Thus

$$P(D) = P((A^c \cap B^c \cap C^c)^c) = 1 - \frac{5}{12} = \frac{7}{12} = 58.3\%$$

4.31. Consider the data in Problem 4.30. (*a*) Find the probability p that exactly one of them hits the target. (*b*) If the target is hit only once, find the probability p that it was the first man A.

(*a*) Let E be the event that exactly one of them hit the target. Then

$$E = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$$

That is, if only one man hit the target then it was only A, $A \cap B^c \cap C^c$, or only B, $A^c \cap B \cap C^c$, or only C, $A^c \cap B^c \cap C$. These three events are mutually exclusive. Thus, we obtain (using Problem 4.79)

$$p = P(E) = P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C)$$

$$= \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{1}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{12} + \frac{5}{36} + \frac{5}{24} = \frac{31}{72} = 43.1\%$$

(*b*) We seek $P(A \mid E)$, the probability that A hit the target given that only one man hit the target. Now $A \cap E = A \cap B^c \cap C^c$ is the event that only A hit the target. Also, by (*a*), $P(A \cap E) = 1/12$ and $P(E) = 31/72$; hence

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{1/12}{31/72} = \frac{6}{31} = 19.4\%$$

4.32. Let $S = \{a, b, c, d\}$ be an equiprobable space; hence each elementary event has probability 1/4. Consider the events:

$$A = \{a, d\}, \quad B = \{b\ d\}, \quad C = \{c, d\}$$

(*a*) Show that A, B, C are pairwise independent.

(*b*) Show that A, B, C are not independent.

(*a*) Here $P(A) = P(B) = P(C) = 1/2$. Since $A \cap B = \{d\}$,

$$P(A \cap B) = P(\{d\}) = \frac{1}{4} = P(A)P(B)$$

Hence A and B are independent. Similarly, A and C are independent and B and C are independent.

(*b*) Here $A \cap B \cap C = \{d\}$, and so $P(A \cap B \cap C) = 1/4$. Therefore

$$P(A)P(B)P(C) = \frac{1}{8} \neq P(A \cap B \cap C)$$

Accordingly, A, B, C are not independent.

4.33. Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is an equiprobable space; hence each elementary event has probability 1/8. Consider the events:

$$A = \{1, 2, 3, 4\}, \qquad B = \{2, 3, 4, 5\}, \qquad C = \{4, 6, 7, 8\}$$

(*a*) Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$.

(*b*) Show that

(i) $P(A \cap B) \neq P(A)P(B)$,

(ii) $P(A \cap C) \neq P(A)P(C)$,

(iii) $P(B \cap C) \neq P(B)P(C)$.

(*a*) Here $P(A) = P(B) = P(C) = 4/8 = 1/2$. Since $A \cap B \cap C = \{4\}$,

$$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$$

(*b*) (i) $A \cap B = \{3, 4, 5\}$, so $P(A \cap B) = 3/8$. But $P(A)P(B) = 1/4$; hence $P(A \cap B) \neq P(A)P(B)$.

(ii) $A \cap C = \{4\}$, so $P(A \cap C) = 1/8$. But $P(A)P(C) = 1/4$; hence $P(A \cap C) \neq P(A)P(C)$.

(iii) $B \cap C = \{4\}$, so $P(B \cap C) = 1/8$. But $P(B)P(C) = 1/4$; hence $P(B \cap C) \neq P(B)P(C)$.

4.34. Prove: Suppose A and B are independent events. Then A^c and B^c are independent events.

We need to show that $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$. Let $P(A) = x$ and $P(B) = y$. Then $P(A^c) = 1 - x$ and $P(B^c) = 1 - y$. Since A and B are independent, $P(A \cap B) = P(A) \cdot P(B) = xy$. Thus, by the addition rule (Theorem 3.6),

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = x + y - xy$$

By DeMorgan's law, $(A \cup B)^c = A^c \cap B^c$; hence

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - x - y + xy$$

On the other hand,

$$P(A^c) \cdot P(B^c) = (1 - x)(1 - y) = 1 - x - y + xy$$

Thus, $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$, and so A^c and B^c are independent.

Similarly, one can show that A and B^c, as well as A^c and B, are independent.

INDEPENDENT REPEATED TRIALS

4.35. A fair coin is tossed three times. Find the probability that there will appear:
(*a*) three heads, (*b*) exactly two heads, (*c*) exactly one head, (*d*) no heads.